

# RELATIVISTIC CORRECTIONS TO THE SUNYAEV-ZEL'DOVICH EFFECT

Anthony Challinor<sup>1</sup> & Anthony Lasenby<sup>2</sup>

MRAO, Cavendish Laboratory, Madingley Road, Cambridge, CB3 0HE, U.K.

## ABSTRACT

We present an extension of the Kompaneets equation which allows relativistic effects to be included to any desired order. Using this, we are able to obtain simple analytic forms for the spectral changes due to the Sunyaev-Zel'dovich effect in hot clusters, correct to first and second order in the expansion parameter  $\theta_e = k_B T_e / mc^2$ . These analytic forms agree with previous numerical calculations of the effect based upon the multiple scattering formalism, and are expected to be very accurate over all regions of the CMB spectrum for  $k_B T_e$  up to  $\sim 10$  keV. Our results confirm previous conclusions that the result of including relativistic corrections in the Sunyaev-Zel'dovich effect is a small *reduction* in the amplitude of the effect over the majority of the spectrum: specifically we find  $\Delta T/T = -2y(1 - 17/10 \theta_e + 123/40 \theta_e^2)$  (correct to second-order) in the Rayleigh-Jeans region, where  $y$  is the usual Comptonization parameter. For a typical cluster temperature of 8 keV, this amounts to a correction downwards to the value of the Hubble constant derived using combined X-ray and Rayleigh-Jeans Sunyaev-Zel'dovich information by about 5 percent.

*Subject headings:* cosmology: cosmic microwave background — galaxies: clusters: general — radiative transfer — scattering

## 1. Introduction

Non-relativistic treatments of the Sunyaev-Zel'dovich effect usually employ the Kompaneets equation (Kompaneets 1957) to determine the distortion of the Cosmic Microwave Background (CMB) spectrum. The Kompaneets equation does not, however, include relativistic effects, which may be important for hot clusters where  $k_B T_e \gtrsim 10$  keV. For this

---

<sup>1</sup>A.D.Challinor@mrao.cam.ac.uk

<sup>2</sup>A.N.Lasenby@mrao.cam.ac.uk

reason, and because of the low optical depth of typical clusters, relativistic treatments of the Sunyaev-Zel’dovich effect usually employ a multiple scattering description of the Comptonization process (Wright 1979), (Fabbri 1981), (Taylor & Wright 1989), (Loeb, McKee, & Lahav 1991), (Rephaeli 1995). Including relativistic effects in this procedure gives a complicated expression for the spectral distortion, which is best handled by numerical techniques (see, for example, (Rephaeli 1995)).

In this paper, we show how the Kompaneets equation may be extended to include relativistic effects in a self-consistent manner, allowing the Sunyaev-Zel’dovich effect in hot clusters to be described on the basis of a Kompaneets type equation. The extension of the Kompaneets equation can be carried out to arbitrary orders in relativistic effects, although we shall only consider the lowest order corrections here. The resulting equation conserves photons at every order, and we demonstrate that it is consistent with earlier calculations of the energy transfer rate between the plasma and a Planck distribution of photons (Woodward 1970).

We then consider the application of the generalised Kompaneets equation to the calculation of the Sunyaev-Zel’dovich effect in hot clusters. Simple analytic forms are given for the spectral distortions in the limit of small optical depth, including relativistic effects to second-order. These are in excellent agreement with Rephaeli’s (1995) numerical calculations, which were based on the multiple scattering approach (truncated at one scattering). This lends further support to Fabbri’s observation that the Boltzmann equation can be applied to describe the Sunyaev-Zel’dovich effect in optically thin clusters (Fabbri 1981), despite claims to the contrary (Wright 1979).

The relativistic corrections to the Sunyaev-Zel’dovich effect are of importance in the calculation of the Hubble constant  $H_0$  by the Sunyaev-Zel’dovich route in hot clusters (see, for example, (Lasenby & Jones 1997) for a recent review of the Sunyaev-Zel’dovich route of determining  $H_0$ , and (Saunders 1996) for recent observations). In the Rayleigh-Jeans region, we find that relativistic effects lead to a small decrease in the Sunyaev-Zel’dovich effect, and hence a small reduction in the hitherto determined values of  $H_0$ , in agreement with the conclusions in (Rephaeli & Yankovitch 1997).

We employ natural units,  $c = \hbar = 1$ , except in the discussion of the Sunyaev-Zel’dovich effect in Section 4.

## 2. Extending the Kompaneets Equation

In this paper we shall not consider effects due to the peculiar motion of the cluster (such effects give rise to a kinetic correction to the Sunyaev-Zel'dovich effect). For a comoving cluster, the CMB photon distribution function is isotropic and may be denoted  $n(\omega)$ , where  $\omega$  is the photon energy. The electrons are assumed to be in thermal equilibrium at temperature  $T_e$ , and are described by an isotropic distribution function  $f(E)$ , where  $E$  is the photon energy. The Boltzmann equation describing the evolution of  $n(\omega)$  may be written as (Buchler & Yeuh 1976)

$$\frac{\partial n(\omega)}{\partial t} = -2 \int \frac{d^3 p}{(2\pi)^3} d^3 p' d^3 k' W [n(\omega)(1 + n(\omega'))f(E) - n(\omega')(1 + n(\omega))f(E')], \quad (2-1)$$

where  $W$  is the invariant transition amplitude for Compton scattering of a photon of 4-momentum  $k^\mu$  by an electron (of charge  $e$  and mass  $m$ ) with 4-momentum  $p^\mu$ , to a photon momentum  $k'^\mu$  and an electron momentum  $p'^\mu$  (Berestetskii, Lifshitz, & Pitaevskii 1982):

$$W = \frac{(e^2/4\pi)^2 \bar{X}}{2\omega\omega'EE'} \delta^4(p + k - p' - k') \quad (2-2)$$

$$\bar{X} \equiv 4m^4 \left( \frac{1}{\kappa} + \frac{1}{\kappa'} \right)^2 - 4m^2 \left( \frac{1}{\kappa} + \frac{1}{\kappa'} \right) - \left( \frac{\kappa}{\kappa'} + \frac{\kappa'}{\kappa} \right), \quad (2-3)$$

with  $\kappa \equiv -2p^\mu k_\mu$  and  $\kappa' \equiv 2p'^\mu k'_\mu$ . In equation (2-1), we have assumed that electron degeneracy effects may be ignored.

The electrons are described by a relativistic Fermi distribution. Since we are ignoring degeneracy effects, we have

$$f(E) \approx e^{-(E - \mu)/k_B T_e}. \quad (2-4)$$

Substituting this form for  $f(E)$  into equation (2-1), and expanding the term in brackets in the integrand in powers of  $\Delta x$ , where

$$x \equiv \frac{\omega}{k_B T_e} \quad (2-5)$$

$$\Delta x \equiv \frac{\omega' - \omega}{k_B T_e}, \quad (2-6)$$

gives a Fokker-Planck expansion

$$\begin{aligned} \frac{\partial n(x)}{\partial t} = & 2 \left( \frac{\partial n}{\partial x} + n(1 + n) \right) I_1 + 2 \left( \frac{\partial^2 n}{\partial x^2} + 2(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right) I_2 \\ & + 2 \left( \frac{\partial^3 n}{\partial x^3} + 3(1 + n) \frac{\partial^2 n}{\partial x^2} + 3(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right) I_3 \\ & + 2 \left( \frac{\partial^4 n}{\partial x^4} + 4(1 + n) \frac{\partial^3 n}{\partial x^3} + 6(1 + n) \frac{\partial^2 n}{\partial x^2} + 4(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right) I_4 + \dots, \end{aligned} \quad (2-7)$$

where

$$I_n \equiv \frac{1}{n!} \int \frac{d^3 p}{(2\pi)^3} d^3 p' d^3 k' W f(E) (\Delta x)^n, \quad (2-8)$$

which does not depend on  $n(\omega)$ .

The calculation of the  $I_n$  may be performed by expanding the integrand in powers of  $p/m$  and  $\omega/m$ . The factor  $f(E)$  is handled by the expansion

$$f(E) \approx e^{(\mu - m)/k_B T_e} e^{-u} \left( 1 + \frac{1}{2} \theta_e u^2 + \frac{1}{8} \theta_e^2 u^3 (u - 4) + \dots \right), \quad (2-9)$$

where

$$u \equiv \frac{p^2}{2mk_B T_e} \quad (2-10)$$

$$\theta_e \equiv \frac{k_B T_e}{m}, \quad (2-11)$$

and the chemical potential  $\mu$  may be eliminated by introducing the electron number density  $N_e$ , which evaluates to

$$N_e = \frac{m\sqrt{2m}}{\pi^2} \Gamma(3/2) (k_B T_e)^{\frac{3}{2}} e^{(\mu - m)/k_B T_e} \left( 1 + \frac{15}{8} \theta_e + \frac{105}{128} \theta_e^2 + \mathcal{O}(\theta_e^3) \right). \quad (2-12)$$

Note that equation (2-12) is an asymptotic expansion about  $\theta_e = 0$  of the result

$$N_e = \frac{m^3}{\pi^2} \theta_e K_2(1/\theta_e) e^{\mu/k_B T_e}, \quad (2-13)$$

where  $K_2(x)$  is a modified Bessel function, which suggests that our series expansions of the  $I_n$  will only be asymptotic series. The calculations of the  $I_n$  are ideally suited to symbolic computer algebra packages (we use Maple). Expressing the results in terms of  $\theta_e$ ,  $x$ , and the Thomson cross section  $\sigma_T$ , we find

$$\begin{aligned} I_1 &= \frac{1}{2} \sigma_T N_e \theta_e x \left( (4 - x) + \theta_e \left( 10 - \frac{47}{2} x + \frac{21}{5} x^2 \right) \right) + \mathcal{O}(\theta_e^3) \\ I_2 &= \frac{1}{2} \sigma_T N_e \theta_e x^2 \left( 1 + \theta_e \left( \frac{47}{2} - \frac{63}{5} x + \frac{7}{10} x^2 \right) \right) + \mathcal{O}(\theta_e^3) \\ I_3 &= \frac{1}{2} \sigma_T N_e \theta_e x^3 \left( \frac{7}{5} (6 - x) \theta_e \right) + \mathcal{O}(\theta_e^3) \\ I_4 &= \frac{1}{2} \sigma_T N_e \theta_e x^4 (7 \theta_e) + \mathcal{O}(\theta_e^3). \end{aligned} \quad (2-14)$$

For  $n > 4$ ,  $I_n$  is third-order or higher in  $\theta_e$ . For CMB photons passing through a cluster at redshift  $z$ , we have  $\bar{x} \simeq 6.2 \times 10^{-4} (1 + z)/k_B T_e$ , where  $\bar{x}$  is the average of  $x$  over a Planck distribution and  $k_B T_e$  expressed in eV. The electron temperature is typically  $\lesssim 10$  keV, so that  $x \ll 1$ .

Substituting these expansions for the  $I_n$  into the series (2-7), we develop an expansion of  $\partial n/\partial t$  in  $\theta_e$ . In this paper, we shall only be concerned with the lowest order relativistic corrections, and start by retaining all terms up to  $O(\theta_e^2)$ . To include all such terms consistently, it is necessary to retain only the first four terms in the series (2-7). A lengthy calculation gives the result

$$\frac{\partial n(x)}{\partial t} = \sigma_T N_e \theta_e \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 j(x) \right), \quad (2-15)$$

where the current  $j(x)$  is given by

$$\begin{aligned} j(x) = & x^2 \left[ \left( \frac{\partial n}{\partial x} + n(1+n) \right) + \theta_e \left[ \frac{5}{2} \left( \frac{\partial n}{\partial x} + n(1+n) \right) + \frac{21}{5} x \frac{\partial}{\partial x} \left( \frac{\partial n}{\partial x} + n(1+n) \right) \right. \right. \\ & \left. \left. + \frac{7}{10} x^2 \left( \frac{\partial^3 n}{\partial x^3} + 2 \frac{\partial^2 n}{\partial x^2} (1+2n) + \frac{\partial n}{\partial x} \left( 1 - 2 \frac{\partial n}{\partial x} \right) \right) \right] + O(\theta_e^2) \right]. \end{aligned} \quad (2-16)$$

The zero-order term in equation (2-16) is just that term which usually appears in the Kompaneets equation (Kompaneets 1957). The  $O(\theta_e)$  term is the lowest-order relativistic correction to the current. The form of equation (2-15) ensures conservation of the total number of photons, which is true for each order in  $\theta_e$ . A similar expression has been derived independently by Stebbins (1997) using non-covariant methods, although he only considers the lowest-order terms in  $\omega/m$ .

This derivation makes it clear that trying to include relativistic terms in the framework of the usual Kompaneets equation (Kompaneets 1957), although it would seem natural to retain only terms in  $I_1$  and  $I_2$ , would be incorrect, since it would not obey photon conservation. To be specific, if only  $I_1$  and  $I_2$  are retained, then photon conservation requires that the current be given by

$$j(x) = 2 \frac{I_2}{\sigma_T N_e \theta_e} \left( \frac{\partial n}{\partial x} + n(1+n) \right), \quad (2-17)$$

and consistency with the expansion of the Boltzmann equation (2-7) requires that

$$I_1 = \frac{\partial I_2}{\partial x} + 2 \frac{I_2}{x} - I_2. \quad (2-18)$$

Calculating  $I_2$  to  $O(\theta_e^2)$  in the limit of small  $x$ , gives the current  $j'(x)$  (in the limit of small  $x$ ) where

$$j'(x) = x^2 \left( \frac{\partial n}{\partial x} + n(1+n) \right) \left( 1 + \frac{47}{2} \theta_e \right), \quad (2-19)$$

which amounts to a relativistic correction to the cross section appearing in the usual Kompaneets equation. This result is incorrect, since the approach is not self-consistent to the order that the answer is quoted. To see this, one need only calculate  $I_1$  using equation (2-18) and

the expression for  $I_2$  from equation (2-14). The expression for  $I_1$  obtained by this procedure only agrees with the direct calculation equation (2-14) to  $O(\theta_e)$ , not to  $O(\theta_e^2)$ . One obtains a better approximation to the true current (eq. (2-16)) by calculating  $I_1$  directly and then using equation (2-18) to calculate an effective value of  $I_2$ , which may then be used to deduce the current. This procedure reproduces the first term in the  $O(\theta_e)$  correction in  $j(x)$ , in the limit of small  $x$ , and is similar to that employed by Fabbri (1981) in obtaining his equation (11). However, it is only the full expression equation (2-16) which is consistent with the original Boltzmann equation (2-1) to  $O(\theta_e^2)$ .

### 3. Rate of Energy Transfer

As a check on the consistency of equation (2-16) with existing results in the literature, we calculate the energy transfer rate between the electrons and a Planck distribution of photons at temperature  $T_r$ .

Multiplying the continuity equation (2-15) by  $x^3$  and integrating, we find

$$\frac{\partial}{\partial t} \int_0^\infty x^3 n(x) dx = -\sigma_T N_e \theta_e \int_0^\infty x^2 j(x) dx. \quad (3-1)$$

The left-hand side of equation (3-1) is proportional to the rate at which the photons are gaining energy per unit volume, denoted  $dE/dt$ . Substituting a Planck distribution for  $n(\omega)$  in equation (2-16) and integrating gives the result

$$\frac{dE_r}{dt} = 4E_r N_e \sigma_T (\theta_e - \theta_r) \left( 1 + \frac{5}{2} \theta_e - 21 \frac{\zeta(6)}{\zeta(4)} \theta_r + O(\theta^2) \right), \quad (3-2)$$

where

$$\theta_r \equiv \frac{k_B T_r}{m}, \quad (3-3)$$

and  $\zeta(x)$  is the Riemann Zeta function.

This result may be compared with a direct evaluation of the energy transfer, obtained by multiplying the transition rate  $W$  by the energy transfer  $\omega' - \omega$ , and integrating over all collisions:

$$\frac{dE_r}{dt} = 4 \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} d^3 p d^3 p' W f(E) n(\omega) [1 + n(\omega')] (\omega' - \omega). \quad (3-4)$$

The integral may be evaluated by a consistent expansion of the integrand. A lengthy calculation gives the result (3-2), which also agrees with the non-covariant calculation of Woodward (1970) (who gives higher-order corrections also). Note that if we had considered only

$I_1$  and  $I_2$ , then using  $j'(x)$  given by equation (2-19), we would have obtained (in the limit of small  $\theta_r$ )

$$\frac{dE_r}{dt} = 4E_r N_e \sigma_T (\theta_e - \theta_r) \left( 1 + \frac{47}{2} \theta_e + O(\theta_e^2) \right), \quad (3-5)$$

which grossly overestimates the importance of the relativistic corrections.

#### 4. The Sunyaev-Zel'dovich Effect

In this section we apply the generalised Kompaneets equation (to first-order in relativistic corrections) to the calculation of the Sunyaev-Zel'dovich effect in optically thin clusters. We consider higher-order effects in the next section.

Following the standard assumptions, we assume that the optical depth is sufficiently small that the spectral distortions are small. In this limit, we may solve equation (2-15) iteratively. The lowest order solution is obtained by substituting the initial photon distribution  $n_0(x)$  into the current (eq. (2-16)). The integral over time is then trivial, and may be replaced by an integral along the line of sight through the cluster, giving

$$\Delta n(x) = \frac{y}{x^2} \frac{\partial}{\partial x} \left( x^2 j(x) \right), \quad (4-1)$$

where  $j(x)$  is evaluated with  $n_0(x)$ , and

$$y \equiv \sigma_T \int N_e \theta_e dl, \quad (4-2)$$

where the integral is taken along the line of sight through the cluster.

For the CMB we take the initial (undistorted) photon distribution to be Planckian with temperature  $T_0$ :

$$n_0(x) = \frac{1}{e^{\alpha x} - 1}, \quad (4-3)$$

where  $\alpha \equiv T_e/T_0$  is the (large) ratio of electron temperature to the CMB temperature. Evaluating equation (4-1) in the limit of large  $\alpha$ , we find the following fractional distortion:

$$\begin{aligned} \frac{\Delta n(X)}{n(X)} &= \frac{y X e^X}{e^X - 1} \left[ X \coth\left(\frac{1}{2}X\right) - 4 + \theta_e \left[ -10 + \frac{47}{2} X \coth\left(\frac{1}{2}X\right) - \frac{42}{5} X^2 \coth^2\left(\frac{1}{2}X\right) \right. \right. \\ &\quad \left. \left. + \frac{7}{10} X^3 \coth^3\left(\frac{1}{2}X\right) + \frac{7X^2}{5 \sinh^2\left(\frac{1}{2}X\right)} \left( X \coth\left(\frac{1}{2}X\right) - 3 \right) \right] \right], \end{aligned} \quad (4-4)$$

correct to first-order in relativistic effects, where

$$X \equiv \frac{\hbar \omega}{k_B T_0}. \quad (4-5)$$

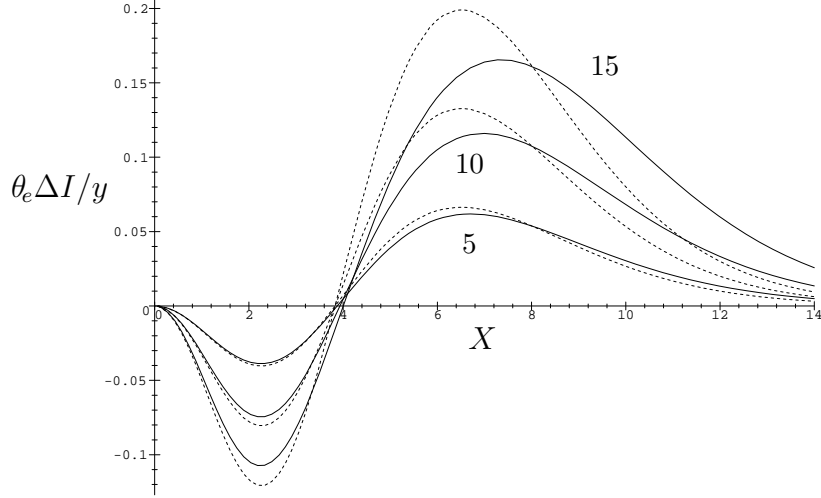


Fig. 1.— The intensity change  $\theta_e \Delta I / y$  (in units of  $2(k_B T_0)^3 / (hc)^2$ ) plotted against  $X$  for three values of  $k_B T_e$  (in keV). The solid curves are calculated using the first-order correction to the Kompaneets equation, while the dashed lines are calculated from the usual Kompaneets expression.

The first two terms in square brackets in equation (4-4) give the usual non-relativistic Sunyaev-Zel’dovich expression, while the terms proportional to  $\theta_e$  are the lowest order relativistic correction. Equation (4-4) agrees with the result in Stebbins (1997). In the Rayleigh-Jeans limit (small  $X$ ), we find

$$\frac{\Delta n(X)}{n(X)} \simeq -2y \left( 1 - \frac{17}{10} \theta_e + O(\theta_e^2) \right). \quad (4-6)$$

In Figure 1 we plot the change in spectral intensity  $\theta_e \Delta I / y$  as a function of  $X$ , where

$$\Delta I = \frac{X^3}{e^X - 1} \frac{\Delta n}{n}. \quad (4-7)$$

Also plotted in Figure 1 are the non-relativistic predictions made with the standard Kompaneets equation. The curves in Figure 1 are for  $k_B T_e = 5, 10$ , and  $15$  keV, which are the same as the parameters used by Rephaeli (1995) in his Figure 1. His calculations, which were based on the multiple scattering formalism (Wright 1979) and required a numerical analysis, give results in excellent agreement with ours, which only require the use of the simple expression (2-8). This suggests that there is no problem in principle with applying the Boltzmann



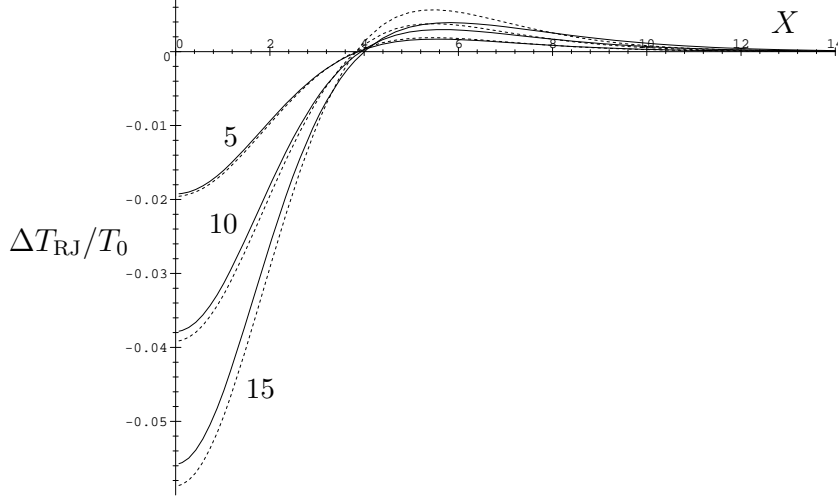


Fig. 2.— The fractional change  $\Delta T_{\text{RJ}}/T_0$  in the Rayleigh-Jeans brightness temperature plotted against  $X$  for three values of  $k_B T_e$  (in keV). The solid curves are calculated using the first-order correction to the Kompaneets equation, while the dashed lines are calculated from the usual Kompaneets expression.

equation to the problem of Comptonization in clusters even though the optical depth may be very small. Similar conclusions were reached by Fabbri (1981), but his demonstration was restricted to low temperature clusters where relativistic effects are not important.

It is clear from Figure 1 that for  $X \lesssim 8$ , the relativistic corrections lead to a reduction in the magnitude of the intensity change, compared to the non-relativistic prediction. This in turn leads to a reduction in the inferred value of the Hubble constant determined by the Sunyaev-Zel’dovich route.

In Figure 2 we plot the fractional change in the Rayleigh-Jeans brightness temperature  $\Delta T_{\text{RJ}}/T_0$  (divided by  $y/\theta_e$ ), where

$$\frac{\Delta T_{\text{RJ}}}{T_0} = \frac{X}{e^X - 1} \frac{\Delta n}{n}, \quad (4-8)$$

for the same  $\theta_e$  as in Figure 1. The relativistic corrections to the change in the Rayleigh-Jeans brightness temperature are significant even at low frequency, unlike the corrections to the intensity, where relativistic corrections are small in the Rayleigh-Jeans part of the spectrum.

## 5. Higher-order Effects

We have found that for  $k_B T_e \gtrsim 10$  keV the second-order relativistic effects make a significant contribution to the spectral distortion, while third-order effects are only significant for  $k_B T_e \gtrsim 15$  keV.

These calculations require a straightforward extension of the method of Section 2 to include terms at  $O(\theta_e^3)$  (for second-order relativistic effects). For the calculation to  $O(\theta_e^3)$ , it is necessary to retain the first six terms of the series (2-7), and to calculate  $I_1$  through  $I_6$  to  $O(\theta_e^3)$ . The first iteration of equation (2-15) for  $T_e \gg T_0$  gives the following next order (in  $\theta_e$ ) correction to  $\Delta n/n$ :

$$\begin{aligned} \left( \frac{\Delta n(X)}{n(X)} \right)^{(2)} &= \theta_e^2 \frac{y X e^X}{e^X - 1} \left[ -\frac{15}{2} + \frac{1023}{8} X \coth\left(\frac{1}{2}X\right) - \frac{868}{5} X^2 \coth^2\left(\frac{1}{2}X\right) \right. \\ &\quad \left. + \frac{329}{5} X^3 \coth^3\left(\frac{1}{2}X\right) - \frac{44}{5} X^4 \coth^4\left(\frac{1}{2}X\right) + \frac{11}{30} X^5 \coth^5\left(\frac{1}{2}X\right) \right. \\ &\quad \left. + \frac{X^2}{30 \sinh^2\left(\frac{1}{2}X\right)} \left( -2604 + 3948 X \coth\left(\frac{1}{2}X\right) - 1452 X^2 \coth^2\left(\frac{1}{2}X\right) + 143 X^3 \coth^3\left(\frac{1}{2}X\right) \right) \right. \\ &\quad \left. + \frac{X^4}{60 \sinh^4\left(\frac{1}{2}X\right)} \left( -528 + 187 X \coth\left(\frac{1}{2}X\right) \right) \right]. \end{aligned} \quad (5-1)$$

In the Rayleigh-Jeans limit, we find

$$\frac{\Delta n(X)}{n(X)} \simeq -2y \left( 1 - \frac{17}{10} \theta_e + \frac{123}{40} \theta_e^2 + O(\theta_e^3) \right). \quad (5-2)$$

In Figure 3 we compare the spectrum of  $\Delta I$  calculated with equation (4-4) to the spectrum with the correction (5-1) included, for  $k_B T_e = 5, 10$  and  $15$  keV ( $\theta_e \approx 0.01, 0.02$  and  $0.03$  respectively). In each case, the second-order relativistic effects are not significant in the Rayleigh-Jeans part of the spectrum. This is to be expected from inspection of equation (5-2), where the  $\theta_e^2$  term is clearly insignificant for the values of  $\theta_e$  considered. For  $k_B T_e = 5$  keV, the second-order effects are insignificant over the entire spectrum. However, for  $k_B T_e \gtrsim 10$  keV, the second-order effects make a significant contribution to the relativistic correction to the Kompaneets based prediction outside the Rayleigh-Jeans region. We have verified that the third-order corrections are negligible over the entire spectrum for  $k_B T_e \simeq 10$  keV. This is confirmed by a comparison of the curves in Figure 3 with the points which are the results of a direct Monte-Carlo evaluation of the Boltzmann collision integral with  $n(\omega)$  given by the Planck distribution  $n_0(x)$  (Gull & Garrett 1997). The second-order effects should be included in the analysis of high frequency data for hot clusters. The magnitude of the second-order correction to the Sunyaev-Zel'dovich result for the rather mild values of  $\theta_e$  considered here,

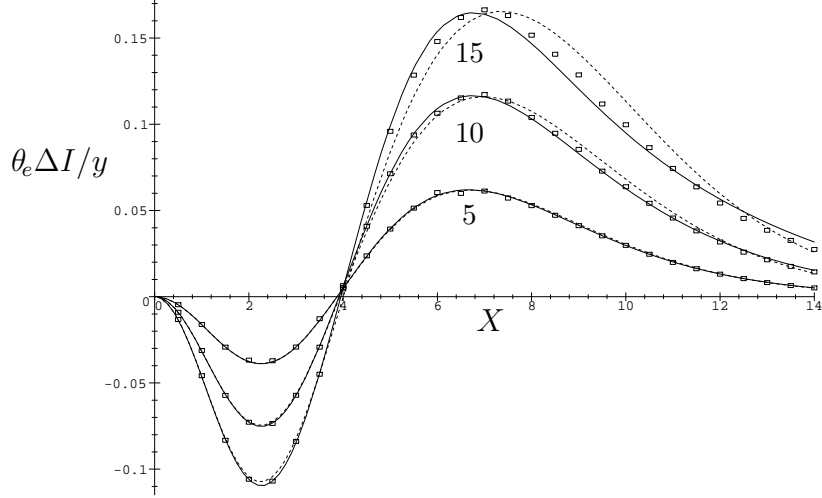


Fig. 3.— The intensity change  $\theta_e \Delta I / y$  (in units of  $2(k_B T_0)^3 / (hc)^2$ ) plotted against  $X$  for three values of  $k_B T_e$  (in keV). The solid curves are calculated using the second-order correction to the Kompaneets equation, while the dashed lines are calculated from the first-order correction. The points are the results of a Monte-Carlo evaluation of the Boltzmann collision integral.

is symptomatic of the asymptotic nature of the series expansion of  $\partial n / \partial t$  in  $\theta_e$ . However, for the majority of clusters considered in Sunyaev-Zel’dovich analyses, the inclusion of the first two relativistic corrections should be sufficient, particularly for experiments working in the Rayleigh-Jeans region of the spectrum.

### 5.1. The Crossover Frequency

The accurate determination of the crossover frequency  $X_0$  (where the thermal component of the spectral distortion vanishes) is essential for reliable subtraction of the kinematic contribution to the Sunyaev-Zel’dovich effect (Rephaeli 1995). In Figure 4 we plot the crossover frequency as a function of  $k_B T_e$ , with the first three relativistic corrections included. For  $k_B T_e \lesssim 20$  keV we find that  $X_0$  is well approximated by the linear relation

$$X_0 \simeq 3.83(1 + 1.13\theta_e). \quad (5-3)$$

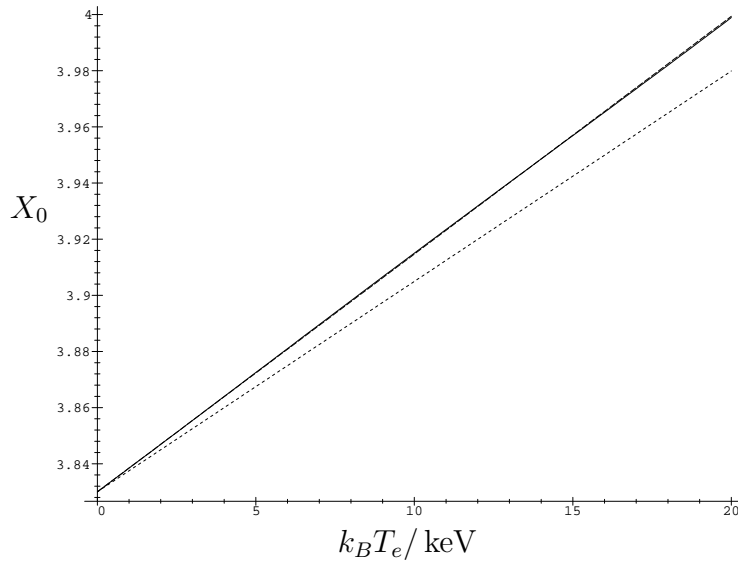


Fig. 4.— The crossover frequency  $X_0$  plotted against  $k_B T_e$ . The solid line is calculated with the inclusion of third-order corrections to the Kompaneets equation. The upper dotted line is a linear fit to the solid line with  $X_0 = 3.83(1 + 1.13\theta_e)$ , while the lower dotted line is the linear fit given in (Rephaeli 1995):  $X_0 = 3.83(1 + \theta_e)$ .

For comparison, Rephaeli (1995), found  $X_0$  to be approximated by  $X_0 \simeq 3.83(1 + \theta_e)$  in the interval  $k_B T_e = 1\text{--}50$  keV, while Fabbri (1981) found  $X_0 \simeq 3.83(1 + 1.1\theta_e)$  for  $k_B T_e \lesssim 150$  keV. It is clear that our calculation favours Fabbri’s expression. For  $k_B T_e \gtrsim 20$  keV,  $X_0$  calculated with the first three relativistic corrections departs from the linear prediction (5-3). However, we do not regard this as indicative of a breakdown of the linear approximation, since it is clear from Figure 3 that the inclusion of higher-order terms may have a significant effect on the value of the crossover frequency.

## 6. Conclusion

We have shown how the Kompaneets equation may be generalised to include relativistic effects in a self-consistent manner. The resulting equation guarantees photon conservation at each order and is in agreement with direct calculations of the energy transfer rate between the plasma and a Planckian distribution of photons.

We have applied this formalism to the calculation of the Sunyaev-Zel’dovich effect in optically thin clusters. We presented simple analytic expressions for the first two relativistic

corrections to the usual Kompaneets based expression for the spectral distortion, which are in excellent agreement with the numerical calculations of Rephaeli (1995) for electron temperatures  $\lesssim 10$  keV. This provides further evidence that the low optical depth of clusters does not forbid the application of the Boltzmann equation to the calculation of the Sunyaev-Zel’dovich effect (Fabbri 1981). The asymptotic nature of the expansion of  $\partial n/\partial t$  in  $\theta_e$  requires the inclusion of higher-order corrections to calculate the effect in hotter clusters in the Wien region of the spectrum. While the calculation of higher-order corrections is not problematic, the bad convergence properties of the series means that ultimately one must resort to a numerical calculation of the collision integral (Corman 1970) (or employ the multiple scattering formalism (Wright 1979)) to calculate the effect in very hot clusters.

Our calculations support the conclusions reached in (Rephaeli & Yankovitch 1997), that including relativistic effects leads to a small decrease in the value of the Hubble constant, inferred from combined X-ray and Sunyaev-Zel’dovich information. For a cluster temperature of  $\simeq 8$  keV, the reduction in  $H_0$  due to relativistic effects is  $\simeq 5$  percent for measurements made in the Rayleigh-Jeans region.

We would like to express our gratitude to Drs N. Itoh, Y. Kohyama and S. Nozawa, whose preprint (Itoh, Kohyama, & Nozawa 1997) stimulated this research, and to Drs S. Gull and A. Garrett for allowing us to quote the (unpublished) results of a Monte-Carlo simulation.

## REFERENCES

- Berestetskii V. B., Lifshitz E. M., & Pitaevskii L. P. 1982, Quantum Electrodynamics: Landau and Lifshitz Course of Theoretical Physics, second edition (Oxford: Pergamon Press plc)
- Buchler J. R., & Yeuh W. R. 1976, ApJ, 210, 440
- Corman E. G. 1970, Phys. Rev. D, 1, 2734
- Fabbri R. 1981, Ap&SS, 77, 529
- Gull S. F., & Garrett A. 1997, in preparation
- Itoh N., Kohyama Y., & Nozawa S. 1997, ApJ, submitted
- Kompaneets A. S. 1957, Soviet Physics JETP, 4, 730

- Lasenby A. N., & Jones M. E. 1997, in *The Extragalactic Distance Scale*, ed. N. Livio et al. (Cambridge: CUP), 76
- Loeb A., McKee C. F., & Lahav O. 1991, *ApJ*, 374, 44
- Rephaeli Y. 1995, *ApJ*, 445, 33
- Rephaeli Y., & Yankovitch D. 1997, *ApJ*, 481, L55
- Saunders R. 1996, in *Microwave Background Anisotropies*, 16th Moriond Meeting, ed. F. R. Bouchet et al. (Editions Frontieres), 377
- Stebbins A. 1997, *ApJ*, submitted
- Taylor G. B., & Wright E. L. 1989, *ApJ*, 339, 619
- Woodward P. 1970, *Phys. Rev. D*, 1, 2731
- Wright E. L. 1979, *ApJ*, 232, 348